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Cardinal functions of Pixley–Roy hyperspaces

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ABSTRACT

Let $\mathcal{F}[X]$ be the Pixley–Roy hyperspace of a regular space X, and let $\mathcal{F}_n[X]=\{F\in\mathcal{F}[X]\colon |F|\leqslant n\}$. For tightness t and supertightness st, we show the following equalities:

(1) $t(\mathcal{F}[X])=\sup\{st(X^n)\colon n\in\mathbb{N}\}$,
(2) $\sup\{t(\mathcal{F}_n[X])\colon n\in\mathbb{N}\}=\sup\{t(X^n)\colon n\in\mathbb{N}\}$.

The first equality answers a question posed in Sakai (1983) [18]. The inequality $\sup\{t(X^n)\colon n\in\mathbb{N}\}\leqslant\sup\{st(X^n)\colon n\in\mathbb{N}\}$ is strict, indeed there is a space Z such that $\sup\{t(X^n)\colon n\in\mathbb{N}\}<\sup\{st(X^n)\colon n\in\mathbb{N}\}$. The discrete countable chain condition and weak Lindelöf property of $\mathcal{F}[X]$ are also investigated.

1. Introduction

All spaces are assumed to be regular. The symbol $\mathbb{N}$ is the set of all positive integers. Unexplained notions and terminology are the same as in [7].

For a space X, let $\mathcal{F}[X]$ be the space of all nonempty finite subsets of X with the *Pixley–Roy topology* [14]: for $A\in\mathcal{F}[X]$ and an open set $U\subset X$, let

$$[A,U]=\{B\in\mathcal{F}[X]\colon A\subset B\subset U\};$$

the family $\{[A,U]\colon A\in\mathcal{F}[X],\ U \text{ open in } X\}$ is a base for the Pixley–Roy topology. It is known that for a T_1-space X, $\mathcal{F}[X]$ is always zero-dimensional, completely regular and every subspace of $\mathcal{F}[X]$ is metacompact: see van Douwen [6]. For each $n\in\mathbb{N}$, we put $\mathcal{F}_n[X]=\{F\in\mathcal{F}[X]\colon |F|\leqslant n\}$. Each $\mathcal{F}_n[X]$ is closed in $\mathcal{F}[X]$, and each $\mathcal{F}_n[X]\setminus\mathcal{F}_{n-1}[X]$ is a discrete space.

The following facts are used in the next section.

Lemma 1.1. *([15, Proposition 1.2]) Let Y be a subspace of a space X. Then $\mathcal{F}[Y]$ is homeomorphic to the closed subspace $\{A\in\mathcal{F}[X]\colon A\subset Y\}$ of $\mathcal{F}[X]$.*

Lemma 1.2. *([11, Theorem 2.8]) For spaces $X_1,\dots,X_k$, $\mathcal{F}[X_1]\times\cdots\times\mathcal{F}[X_k]$ can be embedded as a closed subspace of $\mathcal{F}[X_1\times\cdots\times X_k]$.*

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2. The tightness of $\mathcal{F}[X]$

Definition 2.1. For a space X and a point $x \in X$, let $t(x, X)$ be the smallest cardinal number $\kappa \geqslant \omega$ with the property that if $A \subset X$ and $x \in \overline{A} \setminus A$, then there is a subset $B \subset A$ such that $x \in \overline{B}$ and $|B| \leqslant \kappa$. The cardinal number $t(X) = \sup\{t(x, X)\colon x \in X\}$ is called the *tightness* of X.

Concerning cardinal functions of Pixley–Roy hyperspaces, the following question was posed in [18, Question 2], where $\Psi(X)$ (resp., $\psi_\Delta(X)$) is the closed pseudocharacter (resp., the diagonal degree) of a space X.

Question 2.2. Determine exactly t, Ψ and ψ_Δ on $\mathcal{F}[X]$ in terms of those on X.

Answering this question, Tanaka [20] gave the equalities $\Psi(\mathcal{F}[X]) = \psi_\Delta(\mathcal{F}[X]) = \psi(X)$. In this section, we answer the case of $t(\mathcal{F}[X])$.

For a space X and a point $x \in X$, a family $\mathcal{P}$ of nonempty subsets of X is said to be a *π-network at x* if every neighborhood of x contains some member of $\mathcal{P}$.

Definition 2.3. ([13]) For a space X and a point $x \in X$, let $st(x, X)$ be the smallest cardinal number $\kappa \geqslant \omega$ with the property that if $\mathcal{P}$ is a π-network at x consisting of finite subsets of X, then there is a subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{Q}$ is a π-network at x and $|\mathcal{Q}| \leqslant \kappa$. The cardinal number $st(X) = \sup\{st(x, X)\colon x \in X\}$ is called the *supertightness* of X.

The supertightness of a space X was denoted by $p(X)$ in [13]. Obviously $t(X) \leqslant st(X)$ holds. There is a supercompact Fréchet–Urysohn space Z with $st(Z) = 2^\omega$ [13, Example 2.6].

Theorem 2.4. *For a space X, the equality $t(\mathcal{F}[X]) = \sup\{st(X^n)\colon n \in \mathbb{N}\}$ holds.*

Proof. Assume $t(\mathcal{F}[X]) = \kappa$, and fix an $n \in \mathbb{N}$ and a point $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$. We show $st(\boldsymbol{x}, X^n) \leqslant \kappa$. Let $\mathcal{P}$ be a π-network at $\boldsymbol{x}$ consisting of finite subsets of X^n. We take an open neighborhood U_i of x_i such that $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $A = \{x_1, \ldots, x_n\}$ and $U = U_1 \cup \cdots \cup U_n$. Let

$$\mathcal{D} = \bigl\{F \in [A, U]\colon \text{there is a member } P \in \mathcal{P} \text{ with } P \subset (U_1 \times \cdots \times U_n) \cap F^n\bigr\}.$$

We observe $A \in \overline{\mathcal{D}}$. Take any basic open neighborhood $[A, V]$ of A. Since $(U_1 \cap V) \times \cdots \times (U_n \cap V)$ is an open neighborhood of $\boldsymbol{x}$, there is a member $P \in \mathcal{P}$ with $P \subset (U_1 \cap V) \times \cdots \times (U_n \cap V)$. Let

$$F = A \cup p_1(P) \cup \cdots \cup p_n(P),$$

where p_i is the projection of X^n to the i-th coordinate. Obviously $F \in [A, V] \cap [A, U]$. Since F^n contains P, $P \subset (U_1 \times \cdots \times U_n) \cap F^n$, thus $F \in [A, V] \cap \mathcal{D}$. Since $t(\mathcal{F}[X]) = \kappa$, there is a subfamily $\{F_\alpha\colon \alpha < \kappa\} \subset \mathcal{D}$ such that $A \in \overline{\{F_\alpha\colon \alpha < \kappa\}}$. For each $\alpha < \kappa$, take a member $P_\alpha \in \mathcal{P}$ such that $P_\alpha \subset (U_1 \times \cdots \times U_n) \cap (F_\alpha)^n$. We observe that $\{P_\alpha\colon \alpha < \kappa\}$ is a π-network at $\boldsymbol{x}$. Let $W_1 \times \cdots \times W_n$ be an open neighborhood of $\boldsymbol{x}$, where W_i is an open neighborhood of x_i such that $W_i \subset U_i$, and $W_i = W_j$ if $x_i = x_j$. Take some $\alpha < \kappa$ with $F_\alpha \in [A, W_1 \cup \cdots \cup W_n]$. Then we have

$$P_\alpha \subset (U_1 \times \cdots \times U_n) \cap (F_\alpha)^n \subset (U_1 \times \cdots \times U_n) \cap (W_1 \cup \cdots \cup W_n)^n = W_1 \times \cdots \times W_n.$$

Thus $st(\boldsymbol{x}, X^n) \leqslant \kappa$.

Conversely assume $\sup\{st(X^n)\colon n \in \mathbb{N}\} = \kappa$. We show $t(\mathcal{F}[X]) \leqslant \kappa$. Let $A = \{x_1, \ldots, x_n\} \in \mathcal{F}[X]$ and assume $A \in \overline{\mathcal{A}} \setminus \mathcal{A}$ for $\mathcal{A} \subset \mathcal{F}[X]$. Take an open neighborhood U_i of x_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Since $[A, U_1 \cup \cdots \cup U_n]$ is an open neighborhood of A, we may assume $\mathcal{A} \subset [A, U_1 \cup \cdots \cup U_n]$. Let

$$\mathcal{P} = \bigl\{(U_1 \cap B) \times \cdots \times (U_n \cap B)\colon B \in \mathcal{A}\bigr\}.$$

Obviously each member of $\mathcal{P}$ is nonempty and finite. We observe that $\mathcal{P}$ is a π-network at the point $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$. Let $W_1 \times \cdots \times W_n$ be an open neighborhood of $\boldsymbol{x}$, where $W_i \subset U_i$ $(1 \leqslant i \leqslant n)$. Take a point $B \in [A, W_1 \cup \cdots \cup W_n] \cap \mathcal{A}$. Then

$$(U_1 \cap B) \times \cdots \times (U_n \cap B) = (W_1 \cap B) \times \cdots \times (W_n \cap B) \subset W_1 \times \cdots \times W_n.$$

Thus $\mathcal{P}$ is a π-network at $\boldsymbol{x}$. By $st(\boldsymbol{x}, X^n) \leqslant \kappa$, there is a subfamily $\{B_\alpha\colon \alpha < \kappa\} \subset \mathcal{A}$ such that $\{(U_1 \cap B_\alpha) \times \cdots \times (U_n \cap B_\alpha)\colon \alpha < \kappa\}$ is a π-network at $\boldsymbol{x}$. We observe $A \in \overline{\{B_\alpha\colon \alpha < \kappa\}}$. Take a basic open neighborhood $[A, V]$ of A. Since $(U_1 \cap V) \times \cdots \times (U_n \cap V)$ is an open neighborhood of $\boldsymbol{x}$, there is some $\alpha < \kappa$ such that

$$(U_1 \cap B_\alpha) \times \cdots \times (U_n \cap B_\alpha) \subset (U_1 \cap V) \times \cdots \times (U_n \cap V).$$

Since B_α is contained in $U_1 \cup \cdots \cup U_n$ (remember $\mathcal{A} \subset [A, U_1 \cup \cdots \cup U_n]$),

$$B_\alpha = (U_1 \cap B_\alpha) \cup \cdots \cup (U_n \cap B_\alpha) \subset (U_1 \cap V) \cup \cdots \cup (U_n \cap V) \subset V.$$

Hence $B_\alpha \in [A, V]$. Thus we have $t(A, \mathcal{F}[X]) \leqslant \kappa$. $\square$

Lemma 2.5. *If a family $\{X_n\colon n \in \mathbb{N}\}$ of spaces has the property that $st(X_1 \times \cdots \times X_n) \leqslant \kappa$ for all $n \in \mathbb{N}$, then $st(\prod_{n\in\mathbb{N}} X_n) \leqslant \kappa$.*

Proof. Let $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ and $\mathcal{P}$ be a π-network at $\boldsymbol{x}$ of finite subsets of $\prod_{n\in\mathbb{N}} X_n$. For each $n \in \mathbb{N}$, let $\mathcal{Q}_n = \{p_n(P)\colon P \in \mathcal{P}\}$, where $p_n : \prod_{n\in\mathbb{N}} X_n \to X_1 \times \cdots \times X_n$ be the projection. Then $\mathcal{Q}_n$ is a π-network at $(x_1, \ldots, x_n)$. For each $n \in \mathbb{N}$, take a subfamily $\mathcal{P}_n \subset \mathcal{P}$ such that $|\mathcal{P}_n| \leqslant \kappa$ and $\{p_n(P)\colon P \in \mathcal{P}_n\}$ is a π-network at $(x_1, \ldots, x_n)$. Let $\mathcal{P}' = \bigcup\{\mathcal{P}_n\colon n \in \mathbb{N}\}$, then $|\mathcal{P}'| \leqslant \kappa$ and $\mathcal{P}'$ is a π-network at $\boldsymbol{x}$. $\square$

By Theorem 2.4 and the preceding lemma, we have the following.

Corollary 2.6. *For a space X, the equality $t(\mathcal{F}[X]) = st(X^\omega)$ holds.*

For a space X, let $\mathcal{F}^1[X] = \mathcal{F}[X]$, and let $\mathcal{F}^n[X] = \mathcal{F}[\mathcal{F}^{n-1}[X]]$ for $n \geqslant 2$. The n-times power of $\mathcal{F}[X]$ is denoted by $\mathcal{F}[X]^n$.

Proposition 2.7. *The following statements hold*:

(1) $t(\mathcal{F}[X]) = st(\mathcal{F}[X])$,
(2) $t(\mathcal{F}[X]) = t(\mathcal{F}[X^n])$ *for all* $n \in \mathbb{N}$,
(3) $t(\mathcal{F}[X]) = t(\mathcal{F}[X]^n)$ *for all* $n \in \mathbb{N}$,
(4) $t(\mathcal{F}[X]) = t(\mathcal{F}^n[X])$ *for all* $n \in \mathbb{N}$.

Proof. (1): Assume $t(\mathcal{F}[X]) = \kappa$. Let $A \in \mathcal{F}[X]$ and let $\mathcal{P}$ be a π-network at A consisting of finite subsets of $\mathcal{F}[X]$. Without loss of generality, we may assume $\mathcal{A} \subset [A, X]$ for all $\mathcal{A} \in \mathcal{P}$ (i.e., every member of $\bigcup\{\mathcal{A}\colon \mathcal{A} \in \mathcal{P}\}$ contains A). For each $\mathcal{A} \in \mathcal{P}$, let $F(\mathcal{A}) = \bigcup \mathcal{A}$. Note that for a basic open neighborhood $[A, U]$ of A, $F(\mathcal{A}) \in [A, U]$ if and only if $\mathcal{A} \subset [A, U]$. We observe $A \in \overline{\{F(\mathcal{A})\colon \mathcal{A} \in \mathcal{P}\}}$. Take any basic open neighborhood $[A, U]$ of A. Then $\mathcal{A} \subset [A, U]$ for some $\mathcal{A} \in \mathcal{P}$, hence $F(\mathcal{A}) \in [A, U]$. By $t(\mathcal{F}[X]) = \kappa$, there is a subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $|\mathcal{Q}| \leqslant \kappa$ and $A \in \overline{\{F(\mathcal{A})\colon \mathcal{A} \in \mathcal{Q}\}}$. This implies that $\mathcal{Q}$ is a π-network at A. Thus we have $st(A, \mathcal{F}[X]) \leqslant \kappa$.

(2): Fix an $n \in \mathbb{N}$. Using Lemma 1.1, we immediately have $t(\mathcal{F}[X]) \leqslant t(\mathcal{F}[X^n])$. Conversely let $t(\mathcal{F}[X]) = \kappa$. Then, by Theorem 2.4, the supertightness of every finite power of X is less than or equal to κ, so is the supertightness of every finite power of X^n. By Theorem 2.4, we have $t(\mathcal{F}[X^n]) \leqslant \kappa$.

(3): This follows from the previous statement (2) and Lemma 1.2.

(4): First we show the equality $t(\mathcal{F}[X]) = t(\mathcal{F}^2[X])$. By Theorem 2.4,

$$t\big(\mathcal{F}^2[X]\big) = t\big(\mathcal{F}\big[\mathcal{F}[X]\big]\big) = \sup\big\{st\big(\mathcal{F}[X]^n\big)\colon n \in \mathbb{N}\big\}.$$

Then obviously $\sup\{st(\mathcal{F}[X]^n)\colon n \in \mathbb{N}\} \geqslant st(\mathcal{F}[X]) \geqslant t(\mathcal{F}[X])$. Thus we have $t(\mathcal{F}^2[X]) \geqslant t(\mathcal{F}[X])$. On the other hand, using Lemma 1.2 and the statements (1), (2) in this proposition, we have $\sup\{st(\mathcal{F}[X]^n)\colon n \in \mathbb{N}\} \leqslant \sup\{st(\mathcal{F}[X^n])\colon n \in \mathbb{N}\} = \sup\{t(\mathcal{F}[X^n])\colon n \in \mathbb{N}\} = t(\mathcal{F}[X])$. Thus $t(\mathcal{F}^2[X]) \leqslant t(\mathcal{F}[X])$. In the equality $t(\mathcal{F}^2[X]) = t(\mathcal{F}[X])$, replacing X by $\mathcal{F}[X]$, we have $t(\mathcal{F}^3[X]) = t(\mathcal{F}^2[X])$. Inductively we have $t(\mathcal{F}^n[X]) = t(\mathcal{F}[X])$. $\square$

We denote by $hd(X)$ (resp., $hl(X)$) the hereditary density (resp., hereditary Lindelöf degree) of a space X.

Proposition 2.8. *For a space X, $t(\mathcal{F}[X]) \leqslant \sup\{hd(X^n)\colon n \in \mathbb{N}\}$ holds.*

Proof. Let $\kappa = \sup\{hd(X^n)\colon n \in \mathbb{N}\}$. First we show $st(X) \leqslant \kappa$. Let $x \in X$ and $\mathcal{P}$ be a π-network at x of finite subsets of X. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \{P \in \mathcal{P}\colon |P| = n\}$. For each $P \in \mathcal{P}_n$, put $P = \{x_1(P), \ldots, x_n(P)\}$ and $\boldsymbol{x}(P) = (x_1(P), \ldots, x_n(P)) \in X^n$. By $hd(X^n) \leqslant \kappa$, we can take a subfamily $\mathcal{Q}_n \subset \mathcal{P}_n$ such that $|\mathcal{Q}_n| \leqslant \kappa$ and $\{\boldsymbol{x}(Q)\colon Q \in \mathcal{Q}_n\}$ is dense in $\{\boldsymbol{x}(P)\colon P \in \mathcal{P}_n\}$. We observe that $\bigcup\{\mathcal{Q}_n\colon n \in \mathbb{N}\}$ is a π-network at the point x. Let U be an open neighborhood of x, and take a member $P \in \mathcal{P}$ with $P \subset U$. Assume $P \in \mathcal{P}_n$. Since the n-times product $U \times \cdots \times U$ is a neighborhood of $\boldsymbol{x}(P)$, there is a $Q \in \mathcal{Q}_n$ with $\boldsymbol{x}(Q) \in U \times \cdots \times U$. This implies $Q \subset U$. Thus $st(X) \leqslant \kappa$. Fix any $m \in \mathbb{N}$. Then obviously $\kappa = \sup\{hd((X^m)^n)\colon n \in \mathbb{N}\}$, so we have $st(X^m) \leqslant \kappa$. By Theorem 2.4, $t(\mathcal{F}[X]) \leqslant \kappa$ holds. $\square$

Remark 2.9. For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. Let $l(X)$ be the Lindelöf degree of a space X. In [17, Theorem 2.1], the inequality $\sup\{st(X^n)\colon n \in \mathbb{N}\} \leqslant l(C_p(X))$ was proved for a Tychonoff space X. Moreover, Zenor gave the equality $hl(C_p(X)) = \sup\{hd(X^n)\colon n \in \mathbb{N}\}$ in [23, Theorem 4*]. Hence, for a Tychonoff space X, we have

$$t\big(\mathcal{F}[X]\big) \leqslant l\big(C_p(X)\big) \leqslant hl\big(C_p(X)\big) = \sup\big\{hd\big(X^n\big)\colon n \in \mathbb{N}\big\}.$$

Now we show the second equality.

Lemma 2.10. *Let $k \in \mathbb{N}$ and assume $t(X^k) \leqslant \lambda$. If $x \in X$ and $\mathcal{P}$ is a π-network at x such that $|P| = k$ for all $P \in \mathcal{P}$, then there is a subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $|\mathcal{P}'| \leqslant \lambda$ and $\mathcal{P}'$ is a π-network at x.*

Proof. For each $P \in \mathcal{P}$, let $P = \{x_1(P), \dots, x_k(P)\}$ and $\boldsymbol{x}(P) = (x_1(P), \dots, x_k(P))$. Then the point $(x, \dots, x) \in X^k$ is in the closure of $\{\boldsymbol{x}(P)\colon\ P \in \mathcal{P}\} \subset X^k$. Using $t(X^k) \leqslant \lambda$, we have a subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $|\mathcal{P}'| \leqslant \lambda$ and the point $(x, \dots, x) \in X^k$ is in the closure of $\{\boldsymbol{x}(P)\colon\ P \in \mathcal{P}'\}$. Obviously $\mathcal{P}'$ is a π-network at x. $\square$

Lemma 2.11. *Let $m, k \in \mathbb{N}$ and assume $t(X^{mk}) \leqslant \lambda$. If $\boldsymbol{x} \in X^m$ and $\mathcal{P}$ is a π-network at $\boldsymbol{x}$ in X^m such that $|P| = k$ for all $P \in \mathcal{P}$, then there is a subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $|\mathcal{P}'| \leqslant \lambda$ and $\mathcal{P}'$ is a π-network at $\boldsymbol{x}$.*

Proof. In Lemma 2.10, replace X by X^m. $\square$

Theorem 2.12. *For a space X, the equality $\sup\{t(\mathcal{F}_n[X])\colon\ n \in \mathbb{N}\} = \sup\{t(X^n)\colon\ n \in \mathbb{N}\}$ holds.*

Proof. Assume $\sup\{t(\mathcal{F}_n[X])\colon\ n \in \mathbb{N}\} \leqslant \lambda$. We show $t(X^n) \leqslant \lambda$ for all $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. Let $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$, $Y \subset X^n$ and $\boldsymbol{x} \in \overline{Y} \setminus Y$. Take an open neighborhood U_i of x_i such that $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. We may assume $Y \subset U_1 \times \dots \times U_n$. Let $A = \{x_1, \dots, x_n\}$. For each $\boldsymbol{y} = (y_1, \dots, y_n) \in Y$, we put $F(\boldsymbol{y}) = A \cup \{y_1, \dots, y_n\} \in \mathcal{F}_{2n}[X]$. Let $\mathcal{A} = \{F(\boldsymbol{y})\colon\ \boldsymbol{y} \in Y\}$. We observe $A \in \overline{\mathcal{A}} \setminus \mathcal{A}$. Obviously $A \notin \mathcal{A}$, because of $\boldsymbol{x} \notin Y$. Take a basic open neighborhood $[A, V]$ of A. Since $\boldsymbol{x} \in V^n$, there is a $\boldsymbol{y} = (y_1, \dots, y_n) \in Y \cap V^n$. Then $F(\boldsymbol{y}) \subset V$, thus $A \in \overline{\mathcal{A}}$. Using $t(\mathcal{F}_{2n}[X]) \leqslant \lambda$, we have a subset $Y' \subset Y$ such that $|Y'| \leqslant \lambda$ and $A \in \overline{\{F(\boldsymbol{y})\colon\ \boldsymbol{y} \in Y'\}}$. We observe $\boldsymbol{x} \in \overline{Y'}$. Take a basic open neighborhood $W_1 \times \dots \times W_n$ of $\boldsymbol{x}$, where $W_i \subset U_i$, and $W_i = W_j$ if $x_i = x_j$. By $A \in \overline{\mathcal{A}}$, there is a $\boldsymbol{y} \in Y'$ such that $F(\boldsymbol{y}) \in [A, W_1 \cup \dots \cup W_n]$. Then $\{y_1, \dots, y_n\} \subset W_1 \cup \dots \cup W_n$, and

$$y_i \in U_i \cap (W_1 \cup \dots \cup W_n) = U_i \cap \left(\bigcup \{W_j\colon\ x_j = x_i\}\right) = U_i \cap W_i = W_i.$$

Thus we have $\boldsymbol{y} \in W_1 \times \dots \times W_n$, consequently $t(X^n) \leqslant \lambda$.

Conversely assume $\sup\{t(X^n)\colon\ n \in \mathbb{N}\} \leqslant \lambda$. Since $\mathcal{F}_1[X]$ is discrete, obviously $t(\mathcal{F}_1[X]) \leqslant \lambda$. Fix any $n > 1$, and assume $t(\mathcal{F}_{n-1}[X]) \leqslant \lambda$. We show $t(\mathcal{F}_n[X]) \leqslant \lambda$. Let $A \in \mathcal{F}_n[X]$, $\mathcal{A} \subset \mathcal{F}_n[X]$ and $A \in \overline{\mathcal{A}} \setminus \mathcal{A}$. Since every point in $\mathcal{F}_n[X] \setminus \mathcal{F}_{n-1}[X]$ is isolated in $\mathcal{F}_n[X]$, there is a $1 \leqslant m < n$ such that $A \in \mathcal{F}_m[X] \setminus \mathcal{F}_{m-1}[X]$. If $A \in \overline{\mathcal{A} \cap \mathcal{F}_{n-1}[X]}$, then by $t(\mathcal{F}_{n-1}[X]) \leqslant \lambda$ there is nothing to do. Therefore we may assume $\mathcal{A} \subset \mathcal{F}_n[X] \setminus \mathcal{F}_{n-1}[X]$. Let $A = \{a_1, \dots, a_m\}$ and take an open neighborhood U_i of a_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Considering the basic open neighborhood $[A, U_1 \cup \dots \cup U_m]$ of A, we may assume $\mathcal{A} \subset [A, U_1 \cup \dots \cup U_m]$. Let $k = n - m$. For each $F \in \mathcal{A}$, we define a nonempty finite subset $P(F) \subset X^m$ as follows. Let $F \setminus A = \{x_1(F), \dots, x_k(F)\}$ and let $\psi_F : \{1, \dots, k\} \to \{1, \dots, m\}$ be the map defined by $x_i(F) \in U_{\psi_F(i)}$. For each $x_i(F) \in F \setminus A$, let $\boldsymbol{x}_i(F) = (a_1, \dots, x_i(F), \dots, a_m) \in X^m$, where the $\psi_F(i)$-th coordinate is $x_i(F)$, and the other j-th coordinate is a_j. Let $P(F) = \{\boldsymbol{x}_i(F)\colon\ 1 \leqslant i \leqslant k\} \subset X^m$ and let $\mathcal{P} = \{P(F)\colon\ F \in \mathcal{A}\}$. Note that $|P(F)| = k$ for all $F \in \mathcal{A}$. We observe that $\mathcal{P}$ is a π-network at $(a_1, \dots, a_m)$. Take a basic open neighborhood $V_1 \times \dots \times V_m$ of $(a_1, \dots, a_m)$, where $V_i \subset U_i$ for all i. By $A \in \overline{\mathcal{A}}$, there is an $F \in [A, V_1 \cup \dots \cup V_m] \cap \mathcal{A}$. Then for each $1 \leqslant i \leqslant k$,

$$x_i(F) \in U_{\psi_F(i)} \cap (V_1 \cup \dots \cup V_m) = U_{\psi_F(i)} \cap V_{\psi_F(i)} = V_{\psi_F(i)}.$$

Thus $\boldsymbol{x}_i(F) = (a_1, \dots, x_i(F), \dots, a_m) \in V_1 \times \dots \times V_m$, hence $P(F) \subset V_1 \times \dots \times V_m$. By Lemma 2.11, there is a subfamily $\mathcal{A}' \subset \mathcal{A}$ such that $|\mathcal{A}'| \leqslant \lambda$ and $\{P(F)\colon\ F \in \mathcal{A}'\}$ is a π-network at $(a_1, \dots, a_m)$. We observe $A \in \overline{\mathcal{A}'}$. Take a basic open neighborhood $[A, W]$ of A. Since $(a_1, \dots, a_m) \in W \times \dots \times W \subset X^m$, there is an $F \in \mathcal{A}'$ such that $P(F) = \{\boldsymbol{x}_i(F)\colon\ 1 \leqslant i \leqslant k\} \subset W \times \dots \times W$. Hence $F \setminus A = \{x_1(F), \dots, x_k(F)\} \subset W$. This implies $F \in [A, W]$, consequently $t(\mathcal{F}_n[X]) \leqslant \lambda$. $\square$

Lemma 2.13. *([7, p. 227]) If a family $\{X_n\colon\ n \in \mathbb{N}\}$ of spaces has the property that $t(X_1 \times \dots \times X_n) \leqslant \kappa$ for all $n \in \mathbb{N}$, then $t(\prod_{n \in \mathbb{N}} X_n) \leqslant \kappa$.*

By Theorem 2.12 and the preceding lemma, we have the following.

Corollary 2.14. *For a space X, the equality $\sup\{t(\mathcal{F}_n[X])\colon\ n \in \mathbb{N}\} = t(X^\omega)$ holds.*

Let Z be the supercompact space in [13, Example 2.6]. This space satisfies $t(Z) = \omega$ (indeed, Fréchet–Urysohn) and $st(Z) = 2^\omega$. Since Z is compact, $t(Z^n) = \omega$ for all $n \in \mathbb{N}$ [7, 3.12.8(f)]. Therefore $\sup\{t(\mathcal{F}_n[Z])\colon\ n \in \mathbb{N}\} = \omega$, but $t(\mathcal{F}[Z]) = 2^\omega$. Let S_κ be the quotient space obtained by identifying all limit points of κ many convergent sequences. It is well known that $t(S_\omega \times S_{2^\omega})$ is uncountable. Hence we can see that $t(\mathcal{F}_3[S_{2^\omega}])$ is uncountable.

3. DCCC and CCC of Pixley–Roy hyperspaces

Definition 3.1. A space X satisfies the *discrete countable chain condition* (shortly, *DCCC*) [22] if every discrete family of nonempty open subsets of X is countable. A space X satisfies the *countable chain condition* (shortly, *CCC*) if every pairwise disjoint family of nonempty open subsets of X is countable.

In this section, we investigate some properties concerning DCCC and CCC of Pixley–Roy hyperspaces. For convenience of the readers, first of all we give a diagram of the notions appeared in this section, where *hl* (resp., *hd*) is hereditary Lindelöf degree (resp., hereditary density).

$$
\begin{array}{cccccccccccccc}
\mathcal{F}[X]: & \sigma\text{-centered} & & \rightarrow & & \text{precaliber } \omega_1 & & & \rightarrow & \text{CCC} & \rightarrow & \text{weakly Lindelöf} & \rightarrow & \text{DCCC} \\
 & \updownarrow & & & & \updownarrow & & & & \updownarrow & & \updownarrow & & \updownarrow \\
X: & \text{cosmic} & \rightarrow & (\mathrm{C}) & \rightarrow & (\mathrm{C}') & \rightarrow & (\mathrm{WS}_f) & \rightarrow & ? & \rightarrow & ? & \rightarrow & ? \\
 & & & & & & & \downarrow & & \searrow & & & & \\
 & & & & & & & hl(X^{\omega}) = hd(X^{\omega}) = \omega & & & \text{WS} \rightarrow & hl(X) = hd(X) = \omega & &
\end{array}
$$

Definition 3.2. A space (X,τ) is *σ-centered* if $\tau \setminus \{\emptyset\}$ is the union of countably many centered subfamilies. A space X has *precaliber ω_1* if for every family $\mathcal{U} = \{U_\alpha\colon\ \alpha < \omega_1\}$ of nonempty open subsets of X, there is an uncountable subset $I \subset \omega_1$ such that the family $\{U_\alpha\colon\ \alpha \in I\}$ is centered. A space X is *weakly Lindelöf* if every open cover $\mathcal{U}$ has a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in X.

For an arbitrary space, each implication of "σ-centered $\rightarrow \cdots \rightarrow$ DCCC" holds obviously, or follows from a simple observation. Note that regularity is needed to show "weakly Lindelöf $\rightarrow$ DCCC".

Definition 3.3. A space is *cosmic* if it has a countable network. A space X satisfies *condition* (C) [9, Definition 2] if for every subspace $Y \subset X$ of cardinality ω_1, every open family $\mathcal{U}$ in Y of cardinality ω_1 has a countable network (i.e., there is a countable family $\mathcal{N}$ of subsets of Y such that every member of $\mathcal{U}$ is the union of certain members of $\mathcal{N}$).

Obviously a cosmic space satisfies condition (C). The following two theorems are due to van Douwen, Hajnal and Juhász respectively.

Theorem 3.4. *([6, Lemma 3.2, Theorem 3.3(b)]) A space X is cosmic if and only if $\mathcal{F}[X]$ is σ-centered.*

Theorem 3.5. *([9, Theorems 1, 2]) The following hold:*

(1) *If a space X satisfies condition* (C)*, then $\mathcal{F}[X]$ satisfies CCC.*
(2) *Under MA_{ω_1}, $\mathcal{F}[X]$ satisfies CCC if and only if X satisfies condition* (C)*.*
(3) *Under CH, there is a space X such that $\mathcal{F}[X]$ satisfies CCC, but X does not satisfy condition* (C)*.*

We introduce condition (C') and recall a weakly separated subset.

Definition 3.6. A space X satisfies condition (C') if for every subset $\{x_\alpha\colon\ \alpha < \omega_1\} \subset X$ and a family $\{U_\alpha\colon\ \alpha < \omega_1\}$ of open subsets of X with $x_\alpha \in U_\alpha$, there is an uncountable subset $I \subset \omega_1$ such that $\{x_\alpha\colon\ \alpha \in I\} \subset \bigcap\{U_\alpha\colon\ \alpha \in I\}$. A subset Y of a space X is *weakly separated* [21] if for each point $y \in Y$, one can assign an open neighborhood U_y of y such that for distinct $y, y' \in Y$, $y \notin U_{y'}$ or $y' \notin U_y$ holds. If a space X has no uncountable weakly separated subset, then we say that X satisfies (WS). If no finite power of a space X has an uncountable weakly separated subset, then we say that X satisfies (WS_f).

Condition (C) obviously implies (C'). We can easily see that (C') implies (WS), and that (C') is closed under finite powers. Hence (C') implies (WS_f). It is known that, if a space X satisfies (WS_f), then $\mathcal{F}[X]$ satisfies CCC [12, Theorem]. Moreover, we can easily see that, if $\mathcal{F}[X]$ satisfies CCC, then X satisfies (WS).

Lemma 3.7. *([23, Theorems 3, 3*]) If a family $\{X_n\colon\ n \in \mathbb{N}\}$ of spaces has the property that $X_1 \times \cdots \times X_n$ is hereditarily Lindelöf (resp., hereditarily separable) for all $n \in \mathbb{N}$, then $\prod_{n \in \mathbb{N}} X_n$ is also hereditarily Lindelöf (resp., hereditarily separable).*

A weakly separated subset is a common generalization of a left-separated subset and a right-separated subset. Therefore, if a space satisfies (WS), then it is hereditarily Lindelöf and hereditarily separable. In particular, if $\mathcal{F}[X]$ satisfies CCC, then X is hereditarily Lindelöf and hereditarily separable. If a space X satisfies (WS_f), then every finite power of X is hereditarily Lindelöf and hereditarily separable, hence X^ω is hereditarily Lindelöf and hereditarily separable by Lemma 3.7.

A cover $\mathcal{C}$ of a set X is said to be an *ω-cover* [8] if every finite subset of X is contained in some member of $\mathcal{C}$.

Lemma 3.8. *([8, Proposition]) Every finite power of a space X is Lindelöf if and only if every open ω-cover of X has a countable ω-subcover.*

Lemma 3.9. *Let $\mathcal{U}$ be an open family of a space X. Let $V(\mathcal{U}) = \{F \in \mathcal{F}[X]\colon\ F \subset U \text{ for some } U \in \mathcal{U}\}$, then it is open-and-closed in $\mathcal{F}[X]$.*

Proof. If $F \in V(\mathcal{U})$, then $F \subset U$ for some $U \in \mathcal{U}$. Obviously $[F, U] \subset V(\mathcal{U})$, thus $V(\mathcal{U})$ is open in $\mathcal{F}[X]$. On the other hand, if $F \in \mathcal{F}[X] \setminus V(\mathcal{U})$, then $[F, X] \cap V(\mathcal{U}) = \emptyset$, thus $V(\mathcal{U})$ is closed in $\mathcal{F}[X]$. □

Theorem 3.10. *If $\mathcal{F}[X]$ satisfies DCCC, then the following hold*:

(1) *X is hereditarily Lindelöf, in particular $|X| \leqslant 2^{\omega}$.*
(2) *For finitely many open subsets $U_1, \dots, U_n$ of X, $U_1 \times \cdots \times U_n$ is Lindelöf.*

Proof. First of all we show that every finite power of X is Lindelöf. By Lemma 3.8, we have only to show that every open ω-cover of X has a countable ω-subcover. Let $\mathcal{U} = \{U_\alpha\colon \alpha < \kappa\}$ be an open ω-cover of X. For each $\alpha < \kappa$, let

$$V_\alpha = V\big(\{U_\alpha\}\big) \setminus V\big(\{U_\beta\colon \beta < \alpha\}\big).$$

By Lemma 3.9, each V_α is open-and-closed in $\mathcal{F}[X]$. The family $\{V_\alpha\colon \alpha < \kappa\}$ is a cover of $\mathcal{F}[X]$. Indeed, let $F \in \mathcal{F}[X]$ and put $\gamma = \min\{\alpha < \kappa\colon F \subset U_\alpha\}$, then $F \in V_\gamma$. Moreover, $V_\alpha \cap V_\beta = \emptyset$ if $\alpha < \beta < \kappa$. Indeed, $F \in V_\alpha$ implies $F \subset U_\alpha$, and $F \in V_\beta$ implies $F \setminus U_\alpha \neq \emptyset$, this is a contradiction. Since $\{V_\alpha\colon \alpha < \kappa\}$ is a pairwise disjoint cover consisting of open-and-closed subsets of $\mathcal{F}[X]$, by DCCC of $\mathcal{F}[X]$ the set $\Gamma = \{\alpha < \kappa\colon V_\alpha \neq \emptyset\}$ must be countable. Let $\Gamma = \{\alpha_n\colon n \in \omega\}$. We observe that $\{U_{\alpha_n}\colon n \in \omega\}$ is an ω-cover of X. Let $F \in \mathcal{F}[X]$, then there is an $n \in \omega$ with $F \in V_{\alpha_n}$. This obviously implies $F \subset U_{\alpha_n}$.

To show that X is hereditarily Lindelöf, it suffices that every open subset of X is Lindelöf. Let U be an open subset of X. It is easy to see that $\mathcal{F}[U]$ is homeomorphic to the open-and-closed subset $V(\{U\})$ in $\mathcal{F}[X]$. Hence $\mathcal{F}[U]$ also satisfies DCCC, by the argument in the preceding paragraph, U is Lindelöf. The fact $|X| \leqslant 2^{\omega}$ is well known for hereditarily Lindelöf spaces: see [10, Remark, p. 13].

Let $U_1, \dots, U_n$ be open subsets of X. Since X is hereditarily Lindelöf, every open subset of X is an F_σ-set. Hence $U_1 \times \cdots \times U_n$ is an F_σ-subset of the Lindelöf space X^n, therefore it is Lindelöf. □

Example 3.11. Let X be the two arrows space [7, 3.10.C]. This space X is compact, first-countable, hereditarily Lindelöf and hereditarily separable. Hence X satisfies (1) and (2) in Theorem 3.10. But we show that $\mathcal{F}[X]$ does not satisfy DCCC. For convenience of the readers, we recall the two arrows space. Let $X = ([0,1] \times \{0,1\}) \setminus \{(0,0), (1,1)\}$, consider the order $\prec$ on X defined as follows: $(x,i) \prec (y,j)$ if $x < y$, or $x = y$ and $i = 0$, $j = 1$. The two arrows space is the space X with the order topology induced by $\prec$. In the sequel, to avoid confusion, a point $(r,i) \in X$ is denoted by $\langle r,i\rangle$, and (a,b) stands for an open interval. For each $r \in (0, 1/4)$, let

$$U_r = \big\{\langle r,1\rangle, \langle 1-r,1\rangle\big\} \cup \big\{\langle p,i\rangle\colon p \in (r,1/4) \cup (1-r,1),\ i = 0,1\big\}.$$

Each U_r is open in X. Let $\mathcal{O}_r = [\{\langle r,1\rangle, \langle 1-r,1\rangle\}, U_r]$ for $r \in (0,1/4)$. Obviously $\mathcal{O}_r \cap \mathcal{O}_{r'} = \emptyset$ if $r \neq r'$. Assume that $\{\mathcal{O}_r\colon r \in (0,1/4)\}$ is not discrete at a point $A \in \mathcal{F}[X]$. Since X (equivalently, $\mathcal{F}[X]$) is first-countable, there are $r_n \in (0,1/4)$ and $A_n \in \mathcal{O}_{r_n}$ $(n \in \omega)$ such that $A \subset A_n$ and $A_n \to A$ in $\mathcal{F}[X]$. By $\langle r_n,1\rangle, \langle 1-r_n,1\rangle \in A_n$, there are distinct two points $x, y \in A$ and an infinite subset $J \subset \omega$ such that $\langle r_n,1\rangle \to x$, $\langle 1-r_n,1\rangle \to y$ $(n \in J)$. For simplicity, we may assume $J = \omega$. Then $\{x,y\} \subset A \subset A_n \subset U_{r_n}$ for all $n \in \omega$. Assume $x = \langle p,1\rangle$ for some $p \in [0,1/4)$. By the condition $\langle r_n,1\rangle \to x = \langle p,1\rangle$, $p < r_n$ for all but finitely many $n \in \omega$. Then $x = \langle p,1\rangle \in U_{r_n}$ for only finitely many $n \in \omega$. This is a contradiction. So let $x = \langle p,0\rangle$ for some $p \in (0,1/4]$. Then $r_n < p$ for all but finitely many $n \in \omega$ and $y = \langle 1-p,1\rangle$ holds. This means $y = \langle 1-p,1\rangle \in U_{r_n}$ for only finitely many $n \in \omega$. This is also a contradiction. We conclude that $\{\mathcal{O}_r\colon r \in (0,1/4)\}$ is a discrete family in $\mathcal{F}[X]$.

Daniels [5, Theorem 1A] noted that, if $\mathcal{F}[X]$ is weakly Lindelöf, then every finite power of X is Lindelöf. The statement (2) in Theorem 3.10 is an improvement of Daniels' result.

A space X is said to be *semi-stratifiable* [4] if for each open set $U \subset X$, one can assign a sequence $\{U_n\colon n \in \omega\}$ of closed subsets of X such that (a) $\bigcup\{U_n\colon n \in \omega\} = U$, (b) $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\colon n \in \omega\}$ is the sequence assigned to V. A semi-stratifiable space is obviously perfect (i.e., every open set is F_σ). The product of countably many semi-stratifiable spaces is semi-stratifiable [4, Theorem 2.1].

Corollary 3.12. *If a space X is semi-stratifiable and $\mathcal{F}[X]$ satisfies DCCC, then X^{ω} is hereditarily Lindelöf and hereditarily separable.*

Proof. Fix an $n \in \mathbb{N}$, then X^n is Lindelöf by Theorem 3.10. Since X^n is perfect (because it is semi-stratifiable), it is hereditarily Lindelöf. Moreover since a semi-stratifiable Lindelöf space is hereditarily separable [4, Theorem 2.8], X^n is hereditarily separable. Our conclusion follows from Lemma 3.7. □

Concerning weak Lindelöfness of $\mathcal{F}[X]$, we note the following.

Proposition 3.13. *If $\mathcal{F}[X]$ is weakly Lindelöf, then every closed subset of X is separable. If $t(X) = \omega$ holds additionally, then X is hereditarily separable.*

Proof. Let Y be a closed subset of X. Consider the open cover

$$\big\{\big[\{y\}, X\big]\colon\ y \in Y\big\} \cup \big\{[F, X \setminus Y]\colon\ F \in \mathcal{F}[X],\ F \cap Y = \emptyset\big\}$$

of $\mathcal{F}[X]$. Take countable subsets $\{y_n\colon\ n \in \omega\} \subset Y$ and $\{F_n\colon\ n \in \omega\} \subset \mathcal{F}[X]$ with $F_n \cap Y = \emptyset$ $(n \in \omega)$ such that the union of the family $\{[\{y_n\}, X]\colon\ n \in \omega\} \cup \{[F_n, X \setminus Y]\colon\ n \in \omega\}$ is dense in $\mathcal{F}[X]$. Assume that there is a point $y \in Y \setminus \overline{\{y_n\colon\ n \in \omega\}}$. Take the open set $[\{y\}, G]$, where $G = X \setminus \overline{\{y_n\colon\ n \in \omega\}}$. Obviously $[\{y\}, G] \cap [F_n, X \setminus Y] = \emptyset$, so $[\{y\}, G] \cap [\{y_n\}, X] \neq \emptyset$ for some $n \in \omega$. This implies $y_n \in G$, a contradiction. Hence $\{y_n\colon\ n \in \omega\}$ is dense in Y. Additionally assume $t(X) = \omega$ and let Y be a subset of X. Then $\overline{Y}$ has a countable dense subset $\{y_n\colon\ n \in \omega\}$. For each $n \in \omega$, take a countable set $Y_n \subset Y$ with $y_n \in \overline{Y}_n$. Then $\bigcup\{Y_n\colon\ n \in \omega\}$ is countable and dense in Y. □

The author does not know if there is a non-separable regular space X such that $\mathcal{F}[X]$ satisfies DCCC. But we show that there is such a space among T_2-spaces.

Lemma 3.14. *Assume $2^\omega > \omega_1$. If K is an uncountable compact metric space, and A is a subset of X such that $|A| = \omega_1$, then the set $B = \{x \in K \setminus A\colon\ |A \cap U| = \omega_1$ for any neighborhood U of $x\}$ has cardinality 2^ω.*

Proof. Recall that every uncountable compact metric space has cardinality 2^ω. For each $x \in K \setminus (A \cup B)$, take an open neighborhood U_x of x such that $|A \cap U_x| \leqslant \omega$. Since $K \setminus (A \cup B)$ is Lindelöf, $K \setminus (A \cup B) \subset \bigcup\{U_{x_n}\colon\ n \in \omega\}$ for a countable subset $\{x_n\colon\ n \in \omega\} \subset K \setminus (A \cup B)$. Then

$$\overline{A \setminus \bigcup\{U_{x_n}\colon\ n \in \omega\}} = \Big(A \setminus \bigcup\{U_{x_n}\colon\ n \in \omega\}\Big) \cup B,$$

and it is an uncountable compact metric space. Hence we have $|B| = 2^\omega$. □

Lemma 3.15. *If a space X has cardinality ω_1 and every countable subset of X is closed in X, then $\mathcal{F}[X]$ does not satisfy DCCC.*

Proof. Let $X = \{x_\alpha\colon\ \alpha < \omega_1\}$. Since every countable subset is closed, $U_\alpha = \{x_\beta\colon\ \beta \geqslant \alpha\}$ is open in X. Let $\mathcal{O} = \{[\{x_\alpha\}, U_\alpha]\colon\ \alpha < \omega_1\}$. Obviously $\mathcal{O}$ is pairwise disjoint. Let $F = \{x_{\alpha_1}, \ldots, x_{\alpha_n}\} \in \mathcal{F}[X]$, where $\alpha_1 < \cdots < \alpha_n$, then $F \in [\{x_{\alpha_1}\}, U_{\alpha_1}]$. Thus $\mathcal{O}$ is a cover. Therefore $\mathcal{F}[X]$ does not satisfy DCCC. □

Let $\mathbb{I} = [0,1]$ be the closed unit interval, and τ be the usual topology on $\mathbb{I}$. We consider a finer topology $\tau' = \{U \setminus D\colon\ U \in \tau,\ D$ is a countable subset of $\mathbb{I}\}$ than τ. Obviously $(\mathbb{I}, \tau')$ is a non-separable T_2-space which is not regular. In Proposition 3.13, regularity was not used. Therefore $\mathcal{F}[(\mathbb{I}, \tau')]$ is not weakly Lindelöf.

Theorem 3.16. *The following are equivalent*:

(1) $2^\omega > \omega_1$ *holds,*
(2) $\mathcal{F}[(\mathbb{I}, \tau')]$ *satisfies DCCC.*

Proof. (1) → (2): Let $\mathcal{O}$ be a family of nonempty open subsets of $\mathcal{F}[(\mathbb{I}, \tau')]$ such that $|\mathcal{O}| = \omega_1$. We show that $\mathcal{O}$ is not discrete. We may put $\mathcal{O} = \{[F_\alpha, U_\alpha \setminus D_\alpha]\colon\ \alpha < \omega_1\}$, where $F_\alpha \in \mathcal{F}[(\mathbb{I}, \tau')]$, $U_\alpha \in \tau$, D_α is a countable subset of $\mathbb{I}$ and $F_\alpha \subset U_\alpha \setminus D_\alpha$. Let $\mathcal{B}$ be a countable base for $(\mathbb{I}, \tau)$ closing under finite unions. For each $\alpha < \omega_1$, take $A_\alpha, B_\alpha \in \mathcal{B}$ such that $F_\alpha \subset A_\alpha \subset \overline{A}_\alpha \subset B_\alpha \subset U_\alpha$, where the closure is taken in τ. Then there are an uncountable set $J_1 \subset \omega_1$ and $A, B \in \mathcal{B}$ such that $F_\alpha \subset A \subset \overline{A} \subset B \subset U_\alpha$ for all $\alpha \in J_1$. By Δ-system lemma [7, 2.7.10(c)], there are an uncountable set $J_2 \subset J_1$ and a finite set $R \subset \mathbb{I}$ such that $F_\alpha \cap F_\beta = R$ for distinct $\alpha, \beta \in J_2$. Moreover, there are an uncountable set $J_3 \subset J_2$ and a $k \in \mathbb{N}$ such that $|F_\alpha| = k$ for all $\alpha \in J_3$. From these observations, replacing $U_\alpha (\alpha \in J_3)$ by B, we may put $\mathcal{O} = \{[F_\alpha, B \setminus D_\alpha]\colon\ \alpha < \omega_1\}$ and this family satisfies

(i) $|F_\alpha| = k$ for all $\alpha < \omega_1$,
(ii) $\overline{\bigcup\{F_\alpha\colon\ \alpha < \omega_1\}} \subset B$, where the closure is taken in τ,
(iii) $F_\alpha \cap F_\beta = R$ for distinct $\alpha, \beta < \omega_1$.

Let $|F_\alpha \setminus R| = m$, and put $F_\alpha \setminus R = \{x_{\alpha,1}, \ldots, x_{\alpha,m}\}$ and $\boldsymbol{x}_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,m}) \in \mathbb{I}^m$. Let $X = \{\boldsymbol{x}_\alpha\colon\ \alpha < \omega_1\}$ and let

$$Y = \big\{\boldsymbol{y} \in \mathbb{I}^m \setminus X\colon\ |X \cap W| = \omega_1 \text{ for any neighborhood } W \text{ of } \boldsymbol{y} \text{ in } (\mathbb{I}, \tau)^m\big\}.$$

By Lemma 3.14, $|Y| = 2^\omega$. Hence we can take a point $\boldsymbol{y} = (y_1, \ldots, y_m) \in Y \setminus (\bigcup\{F_\alpha \cup D_\alpha\colon\ \alpha < \omega_1\})^m$. Let $F_{\boldsymbol{y}} = \{y_1, \ldots, y_m\}$. Then obviously $F_{\boldsymbol{y}} \cap (\bigcup\{F_\alpha \cup D_\alpha\colon\ \alpha < \omega_1\}) = \emptyset$, and $F_{\boldsymbol{y}} \subset B$ because of (ii) above. We see that every neighborhood of $F_{\boldsymbol{y}} \cup R$ intersects with uncountably many members of $\mathcal{O}$. Let $[F_{\boldsymbol{y}} \cup R, V \setminus D]$ be a basic open neighborhood of $F_{\boldsymbol{y}} \cup R$, where $V \in \tau$, D is a countable set in $\mathbb{I}$ and $F_{\boldsymbol{y}} \cup R \subset V \setminus D$. Since $\boldsymbol{y} \in Y \cap (V \times \cdots \times V)$ and D is countable, there is an uncountable set $J \subset \omega_1$ such that $F_\alpha \setminus R \subset V$ and $(F_\alpha \setminus R) \cap D = \emptyset$ for all $\alpha \in J$. Then $F_\alpha \subset V \setminus D$ and $F_{\boldsymbol{y}} \cup R \subset B \setminus D_\alpha$ for all $\alpha \in J$. This means $[F_{\boldsymbol{y}} \cup R, V \setminus D] \cap [F_\alpha, B \setminus D_\alpha] \neq \emptyset$ $(\alpha \in J)$. Thus $\mathcal{O}$ is not discrete, so $\mathcal{F}[(\mathbb{I}, \tau')]$ satisfies DCCC.

(2) $\to$ (1): This follows from Lemma 3.15. $\square$

Remark 3.17. In ZFC, there is a non-separable T_1-space X such that $\mathcal{F}[X]$ satisfies DCCC. Let X be a set of cardinality ω_2. We give X the topology $\tau = \{\emptyset\} \cup \{X \setminus D\colon\ D$ is a countable set in $X\}$. DCCC of $\mathcal{F}[X]$ can be proved by the same argument as in Theorem 3.16(1) $\to$ (2).

We give a characterization for $\mathcal{F}[X]$ to have precaliber ω_1.

Theorem 3.18. *For a space X, the following are equivalent*:

(1) *$\mathcal{F}[X]$ has precaliber ω_1*,
(2) *X satisfies condition* (C$'$).

Proof. (1) $\to$ (2): Let $\{x_\alpha\colon\ \alpha < \omega_1\} \subset X$ and $\{U_\alpha\colon\ \alpha < \alpha\}$ be an open family in X with $x_\alpha \in U_\alpha$. Consider the open family $\{[x_\alpha, U_\alpha]\colon\ \alpha < \omega_1\}$ in $\mathcal{F}[X]$. Then, using the condition (1), we have an uncountable subset $I \subset \omega_1$ such that $\{[x_\alpha, U_\alpha]\colon\ \alpha \in I\}$ is centered. Fix any $\alpha \in I$. Then for each $\beta \in I$, $[x_\alpha, U_\alpha] \cap [x_\beta, U_\beta] \neq \emptyset$, hence $x_\alpha \in U_\beta$. Therefore we have $\{x_\alpha\colon\ \alpha \in I\} \subset \bigcap\{U_\alpha\colon\ \alpha \in I\}$.

(2) $\to$ (1): Let $\mathcal{U}$ be a family of cardinality ω_1 consisting of nonempty open subsets of $\mathcal{F}[X]$. We may assume that every member of $\mathcal{U}$ is a basic open set of $\mathcal{F}[X]$. We put $\mathcal{U} = \{[F_\alpha, U_\alpha]\colon\ \alpha < \omega_1\}$, where $F_\alpha \in \mathcal{F}[X]$ and U_α is an open set in X containing F_α. Moreover, we may assume that there is a $k \in \mathbb{N}$ such that $|F_\alpha| = k$ for all $\alpha < \omega_1$. Let $F_\alpha = \{x_{\alpha,i}\colon\ 1 \leqslant i \leqslant k\}$. Applying the condition (2) to $\{x_{\alpha,1}\colon\ \alpha < \omega_1\}$ and $\{U_\alpha\colon\ \alpha < \omega_1\}$, we have an uncountable subset $I_1 \subset \omega_1$ such that $\{x_{\alpha,1}\colon\ \alpha \in I_1\} \subset \bigcap\{U_\alpha\colon\ \alpha \in I_1\}$. Continuing this operation, inductively we have an uncountable subset $I_k \subset I_{k-1}$ such that $\bigcup\{F_\alpha\colon\ \alpha \in I_k\} \subset \bigcap\{U_\alpha\colon\ \alpha \in I_k\}$. Then $\{[F_\alpha, U_\alpha]\colon\ \alpha \in I_k\}$ is centered. $\square$

Corollary 3.19. *If $\mathcal{F}[X]$ has precaliber ω_1, then X^ω is hereditarily Lindelöf and hereditarily separable.*

Recall the diagram above. Using Theorem 3.5(2), we have the following.

Corollary 3.20. *Under MA_{ω_1}, if $\mathcal{F}[X]$ satisfies CCC, then X^ω is hereditarily Lindelöf and hereditarily separable.*

The following questions look interesting.

Question 3.21. Let X be a regular space. If $\mathcal{F}[X]$ satisfies DCCC, then is X (hereditarily) separable? In particular, if L is a Souslin line, then does $\mathcal{F}[L]$ satisfy DCCC?

Question 3.22. Let X be a regular space. If $\mathcal{F}[X]$ satisfies DCCC, then is $\mathcal{F}[X]$ weakly Lindelöf?

Question 3.23. If $\mathcal{F}[X]$ is weakly Lindelöf, then is X hereditarily separable (equivalently, of countable tightness)?

4. An application of Pixley–Roy hyperspaces

We give an application on DCCC of Pixley–Roy hyperspaces. According to [2], a space X is said to be *feebly Lindelöf* if every locally finite family of nonempty open subsets of X is countable, and a space X is said to be *star Lindelöf* if for every open cover $\mathcal{U}$ of X, there is a Lindelöf subspace $L \subset X$ such that $st(L, \mathcal{U}) = X$, where $st(L, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap L \neq \emptyset\}$. For a regular space, DCCC and feebly Lindelöf property are equivalent [22, Theorem 2.6]. A star Lindelöf space is feebly Lindelöf [2, Theorem 2.7]. Alas et al. asked whether a T_4 (= normal T_1) feebly Lindelöf space is star Lindelöf [2, p. 626]. Answering this question, under $2^\omega = 2^{\omega_1}$ Song gave a counterexample [19, Example 2.2]. We show that under $MA + 2^\omega > \omega_1$ (Martin's axiom plus the negation of the continuum hypothesis) there is a T_4 CCC (hence, feebly Lindelöf) metacompact Moore space which is not star Lindelöf.

Lemma 4.1. *([11, Theorem 2.3]) For a space X, $\mathcal{F}[X]$ is the union of countably many closed discrete subspaces if and only if every point of X is G_δ.*

Proposition 4.2. *For a space X, $\mathcal{F}[X]$ is star Lindelöf if and only if X is countable.*

Proof. Assume that $\mathcal{F}[X]$ is star Lindelöf, and consider the open cover $\mathcal{U} = \{[\{x\}, X]\colon\ x \in X\}$ of $\mathcal{F}[X]$. Take a Lindelöf subspace $L \subset \mathcal{F}[X]$ such that $st(L, \mathcal{U}) = \mathcal{F}[X]$. Since a star Lindelöf space is feebly Lindelöf (= DCCC), by Theorem 3.10(1)

X is hereditarily Lindelöf, hence every point of X is G_δ. Therefore, by Lemma 4.1, $\mathcal{F}[X]$ is the union of countably many closed discrete subspaces. This implies that L is countable. Let $L = \{F_n\colon n \in \omega\}$. If $x \in X$, then $\{x\} \in st(L, \mathcal{U})$, so there are a point $y \in X$ and a $k \in \omega$ such that $F_k \in [\{y\}, X]$ and $\{x\} \in [\{y\}, X]$. Then obviously $x = y$, so we have $x \in F_k$. Thus $X = \bigcup\{F_n\colon n \in \omega\}$.

The converse is trivial. □

Lemma 4.3. *([6, Proposition 2.5]) A space X is first-countable if and only if $\mathcal{F}[X]$ is a Moore space.*

Theorem 4.4. *(Przymusiński and Tall [16]) Under $MA + 2^\omega > \omega_1$, if X is a subspace of the real line with $|X| = \omega_1$, then $\mathcal{F}[X]$ is normal.*

Example 4.5. Assume $MA + 2^\omega > \omega_1$, and let X be a subspace of the real line with $|X| = \omega_1$. Then, by Proposition 4.2, Lemma 4.3 and Theorem 4.4, $\mathcal{F}[X]$ is a T_4 CCC (hence, feebly Lindelöf) metacompact Moore space which is not star Lindelöf.

Song's counterexample in [19] is neither CCC, metacompact nor a Moore space, because it contains the space ω_1 with the order topology as an open-and-closed subspace.

Alas et al. asked also whether a first-countable star Lindelöf space is star countable [2, p. 626], where a space X is said to be *star countable* if for every open cover $\mathcal{U}$ of X, there is a countable set $A \subset X$ such that $st(A, \mathcal{U}) = X$. This question was solved in the negative [1, Example 3]. Aiken's counterexample is not pseudocompact. We comment that there is a pseudocompact counterexample. Bell [3, Example 5.1] showed that if a Tychonoff space X is a first-countable, zero-dimensional, locally compact, metaLindelöf, non-compact space in which all nonempty open sets have π-weight 2^ω, then X has a first-countable, metaLindelöf, non-compact pseudocompactification. Let $\mathbb{C}$ be the usual Cantor set in the closed unit interval. Let K be the space $\mathbb{C}^2$ with the topology induced by the lexicographic order on it. Let X be the topological sum of ω many copies of K^ω. Then, by Bell's result above, we have a first-countable, metaLindelöf, non-compact pseudocompactification Y of X. Since Y has a dense σ-compact space X, obviously it is star Lindelöf. On the other hand, since a metaLindelöf star countable space is Lindelöf, Y is not star countable.

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